Massachusetts Institute of Technology
1.200J—Transportation Systems Analysis: Performance and Optimization
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Recitation 8
Unit 4 — Network Flows: Models and Algorithms

1 Algorithm Complexity

1.1 Identifying Complexity

What is the asymptotic complexity of the following methods, in terms of the Big-O notation. Let n be an input which takes only positive integer value.

function methodA(n)
for $j = n : -2 : 1$
disp(j);
end
end
	function methodB(n)
	for $j = 1 : n$
	for $k = n : -1 : n - j + 1$
	disp(k);
	end
	end
	end
		function methodC(n)
		for $j = n : n$
		$k = n$;
		while $k > 1$
		disp(k);
		$k = k/2$;
		end
		end
			function methodD(n)
			for $j = n : -1 : 1$
			for $k = 0 : j - 1$
			methodD2(n);
			end
			end
			end
			
			function methodD2(n)
			$j = 1$
			while $j < n$
			disp(j);
			$j = 3j$;
			end
			end

2 Network Flows

2.1 Leia's Getting a Used Car

Leia Organa has just graduated from Alderaan High School. She will begin her five-year Bachelor-Masters program in Transportation at MIT. As a graduation present, her parents have given her a car fund of $12,000 to help purchase and maintain a certain three-year-old used car for college. Since operating and maintenance costs go up rapidly as the car ages, Leia's parents tell her that she will be welcome to trade in her car on another three-year-old car one or more times during the next four summers if she determines that this would minimize her total net cost. They also inform her that they will give her a new car in five years as a graduation present, so she should definitely plan to trade in her car then. (These are pretty nice parents!)

The table gives the relevant data for each time Leia purchases a three-year-old car. For example, if she trades in her car after two years, the next car will be in ownership year 1 during her junior year, etc.

Year	1	2	3	4	5
Costs ($ thousands)	2	4	5	9	12
Gain ($ thousands)	7	6	2	1	0

When should Leia trade in her car (if at all) during the next four summers to minimize her total net cost of purchasing, operating, and maintaining the cars over her five years at MIT?

(a) Formulate the problem as a Network Flow problem.

(b) Solve for the optimal strategy for Leia.

2.2 Tournament Problems

There are n students in 1.200J. Each of the student plays a game against every other member a total of k games. Assume that every game ends in a win or loss (no draws) and let x_i be the number of wins of team i. Let X be the set of all possible outcome vectors $(x_1, ..., x_n)$.

Given an arbitrary vector $(y_1, ..., y_n)$, we would like to determine whether it belongs to X, that is, whether it is possible tournament outcome vector. Provide a network flow formulation of this problem.

3 Solution

3.1

(a) Here, in the inner loop is decreased by 2 (from n to 2) in each iteration, so the loop runs $n/2$ times. Hence the complexity is $\mathrm{O}(n)$.

(b) In this case there are two nested loops, in the outer loop is incremented by 1 in each iteration, in the inner loop is decreased by 1. The complexity of the outer loop is $\mathrm{O}(n)$ and the inner loop is $\mathrm{O}(n)$. Hence the complexity of the code block is $\mathrm{O}(n^2)$. Alternatively, the total number of times the loop is executed is given by $n + n - 1 + ... + 2 + 1 = n(n-1)/2$. The Big (O) of $n(n-1)/2$ is given by $\mathrm{O}(n^2)$.

(c) In this case there are two nested loops, in the outer loop is incremented by 1 in each iteration, while in the inner loop is divided by 2, so you might think that the complexity is of $\mathrm{O}(n \log n)$. But look closely, in the first loop, the index is initialized to n, so in effect it is executed only once due to which the complexity of the code block becomes $\mathrm{O}(\log n)$.

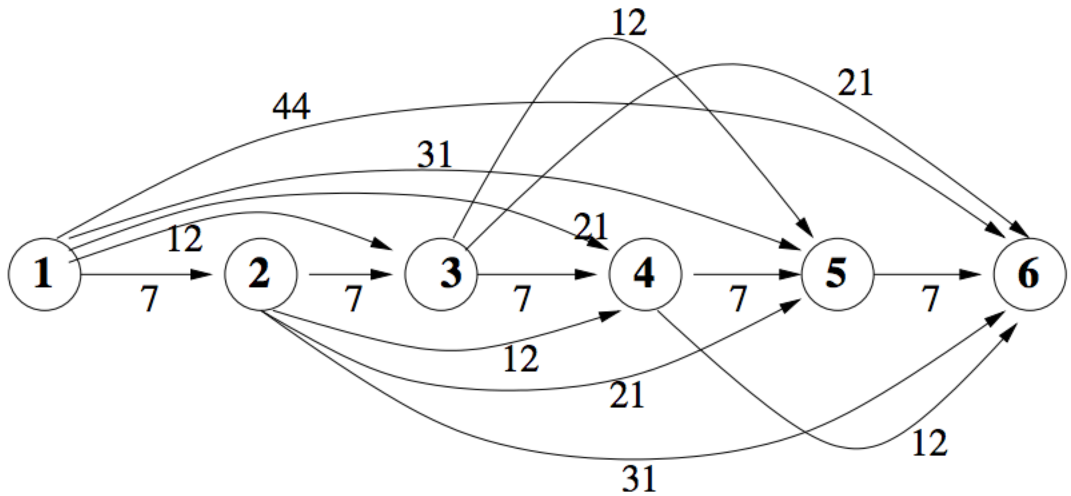
(d) The methodD2 function gets executed $(\log_3 n)$ times. As the loop is executed n times, the number of iterations will be $((\log_3 N) + 2(\log_3 N) + 3(\log_3 N) + ... + N(\log_3 N))$. This will be $n(n-1)(\log_3 n)/2$. The Big(O) of this equation is $n^2(\log n)$. (Note that you can always convert any $\log_a b$ to base 2 by $\log_2 b / \log_2 a$.)

3.2

(a) Consider a directed graph with 6 nodes. Nodes 1 to 5 are associated to the start of each year. The sixth node represents the graduation. For each $i < 6$ and $j > i$, arc (i, j) represents the occurrence that Leia buy a car at the beginning of the i-th year and sell it at the beginning of the j-th year. The cost c_{ij} associated to the arc (i, j) is given by:

$$c_{ij} = a_i + \sum_{k=1}^{j-1} m_k - r_j,$$

where a_j is the price of a new car (equal to \$12,000), m_k is the maintenance cost in the k-th year and r_j is the gain from the sale of the old car. We obtain the following graph:



Any path from 1 to 6 represents a renewal plan; the cost of the path is the cost of the plan. We have to find a shortest path from node 1 to node 6.

(b) To this end we may either apply Dijkstras algorithm stopping as soon as node 6 has been settled, or we may notice that the graph is acyclic. This allows us to solve the problem using a dynamic programming technique. Let $\Pi(j)$ and $N(j)$ be the optimal cost when Leia graduates in Year j and the previous time she got a new car, respectively. We obtain the following values:

(a) $\Pi(1) = 0$;

(b) $\Pi(2) = 7, N(2) = 1$;

(c) $\Pi(3) = 12, N(3) = 1$;

(d) $\Pi(4) = 19, N(4) = 3$;

(e) $\Pi(5) = 24, N(5) = 3$;

(f) $\Pi(6) = 31, N(6) = 5$;

The shortest path (having cost 31) is $1 \to 3 \to 5 \to 6$. In other words, Leia should buy new car every two years. Note that this solution is not unique.

3.3

We introduce nodes $T_1, .., T_n$ that correspond to the different teams. These are supply nodes and node T_i has a supply of x_i, the total number of games won by team i. For every unordered pair i, j of teams, we introduce a node G_{ij}. These are demand nodes, with demand k, the total number of games played between these two teams. Since the total number of games must be equal to the total number of wins, we assume that $\sum_{i=1}^{n} x_i = kn(n-1)/2$.

There are two arcs that come into a node G_{ij}; one from T_i and one from T_j. The flow from T_i to G_{ij} represents the total number of games between teams i and j that were won by team i.

The above constructed network flow problem is feasible if and only if the vector $(y_1, ..., y_n)$ belongs to the set of possible outcome vectors.