Massachusetts Institute of Technology
1.200J—Transportation Systems Analysis: Performance and Optimization
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Recitation 11
Unit 5 — Advanced Stochastic Modeling of Transportation Applications

1 Probability Refresher

X has pdf $\lambda e^{-\lambda x}\mathbb{1}_{x\geq 0}$ for some $\lambda > 0$. Compute the following:

(a) The cdf of X

(b) The expected value of X

(c) The variance of X

(d) The probability that X is greater than its expected value.

2 Inverse Transform Method

(a) Weibull distribution with parameter (α, β) has the following probability density function (pdf)

$$f(x) = \begin{cases} \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}} & \text{if } x \geq 0; \\ 0 & \text{otherwise} \end{cases}$$

Use the inverse transform method to generate samples from the Weibull distribution. Write down the pseudo-code.

(b) Write down the pseudo-code for an algorithm to generate X with the following cdf

$$F(x) = \left(0.4(1 - e^{-2x}) + 0.6(1 - e^{-2\sqrt{x}})\right) \mathbb{1}_{x \geq 0}$$

by using two random variables $U_1, U_2 \sim$ Uniform(0,1).

3 Discrete Event Simulation

(a) Simulate a M/M/1 queueing system that stops admitting new customers after time T. PDFs for interarrival times and service times are f_a and f_s, respectively.

  (i) Define variables and events for the system.
  (ii) Initialization
  (iii) Event 1: Arrival of a customer
  (iv) Event 2: Departure of a customer
  (v) Event 3: Serving customers after system closure

(b) Simulate a M/M/1 queuing system up until N customers getting served.

Solution - Probability Refresher

(a) The key is to integrate the PDF from the lower bound up to x as the definition of CDF is $\mathbb{P}(X \leq x)$.
CDF: $F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} \lambda e^{-\lambda x} \mathbb{1}_{x \geq 0} dx = \int_0^x \lambda e^{-\lambda x}$ for $x \geq 0$. Thus,

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) $\mathbb{E}[X] = \int_0^\infty x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$

(c) $Var[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \int_0^\infty x^2 \lambda e^{-\lambda x} dx - (\frac{1}{\lambda})^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

(d) $\mathbb{P}(X \geq 1/\lambda) = 1 - \mathbb{P}(X \leq 1/\lambda) = 1 - (1 - e^{-1})) = 1/e$ or you can also achieve the same answer by calculating $\int_{1/\lambda}^{\infty} \lambda e^{-\lambda x} dx = e^{-1}$.

Solution - Inverse Transform Method

(a) The inverse transform method relies on the fact that for $X \sim F_X$ we have that $F_X^{-1}(X) \sim Uniform(0, 1)$. For this example, we need to compute the distribution function of the general Weibull distribution and then compute its inverse:

$$F_X(x) = \int_0^x f(y) dy = \alpha\beta \int_0^x y^{\beta-1} e^{-\alpha y^\beta} dy = 1 - e^{-\alpha x^\beta}$$

so that

$$x = \left(-\frac{\log(1 - F_X(x))}{\alpha}\right)^{1/\beta}$$

and we can define:

$$F_X^{-1}(U) = \left(-\frac{\log(1 - U)}{\alpha}\right)^{1/\beta}$$

as $(1 - U)$ is also of Uniform(0,1), we can say that:

$$F_X^{-1}(U) = \left(-\frac{\log(U)}{\alpha}\right)^{1/\beta}$$

Define `Finv(u, alpha, beta)` to be exactly this function, then we can use the following pseudo-code to generate random variables:

```
function [sample] = weibull(alpha, beta)
u = rand(1); % generate a uniform(0,1) r.v.
sample = Finv(u, alpha, beta);
end
```

(b) The CDF:

$$F(x) = \left(0.4(1 - e^{-2x}) + 0.6(1 - e^{-2\sqrt{x}})\right) \mathbb{1}_{x \geq 0}$$

Let $U_1, U_2 \sim$ Uniform(0,1) and $p = 0.4$. We notice that that $(1 - e^{-2x})$ is F_1 of the exponential distribution at rate $\lambda = 2$, while $F_2 = 1 - e^{-2\sqrt{x}}$ is of Weibull distribution we solved in part (b) with $\alpha = 2, \beta = 1/2$.

We found the inverse functions for both distributions:

$$F_1^{-1}(U) = -\frac{\log(U)}{2} \qquad \text{and} \qquad F_2^{-1}(U) = \frac{(\log(U))^2}{4}$$

Note that we will use U_1 to choose which F^{-1} to use, and use U_2 to generate the sample. The pseudo-code:

(a) Generate $U_1, U_2 \sim$ Uniform(0,1).

(b) If $U_1 \geq 0.4$, set $X = -0.5 \ln(U_2)$; otherwise, $X = 0.25(\ln(U_2))^2$.

MATLAB:

```
function [sample] = mixture
u1 = rand(1); u2 = rand(1);
if (u1 <= 0.4) sample = -0.5*log(u2);
else sample =0.25*(log(u2))$^2$; end
end
```