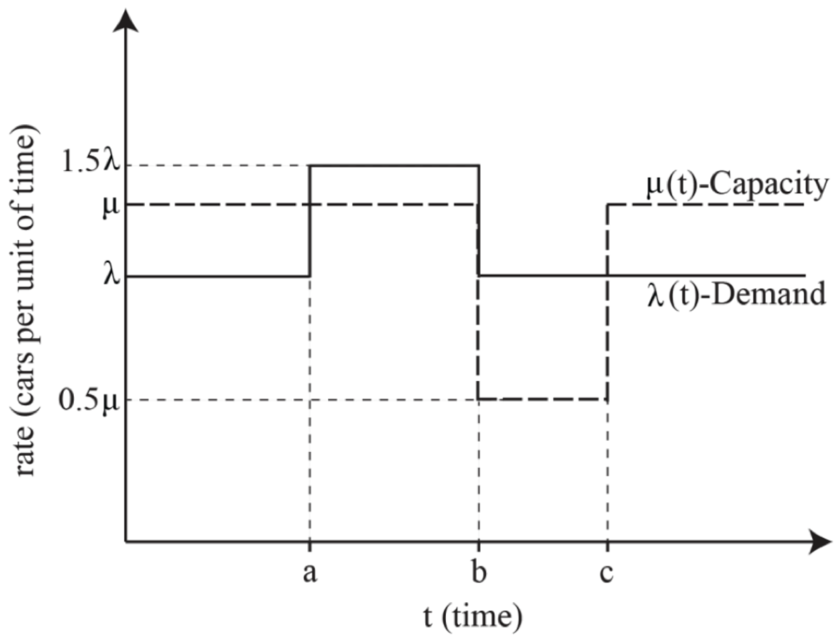
Massachusetts Institute of Technology
1.200J—Transportation Systems Analysis: Performance and Optimization
Fall 2015 — TA: Wichinpong "Park" Sinchaisri

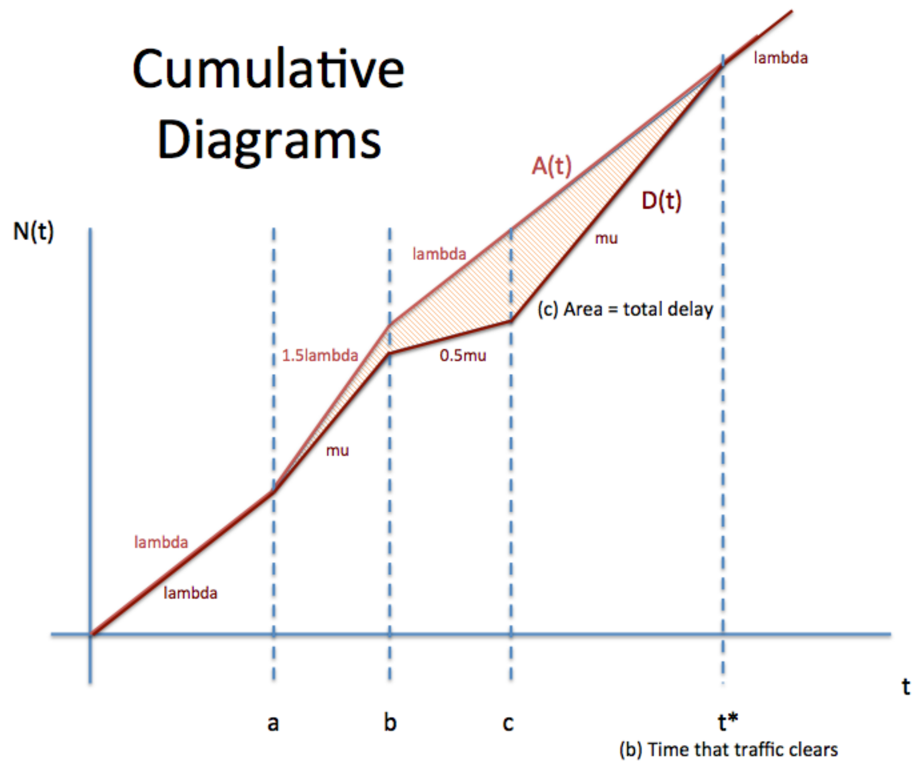
Recitation 1
Unit 1 — Capacity, Delays, and Flow Models

1 Cumulative Diagrams

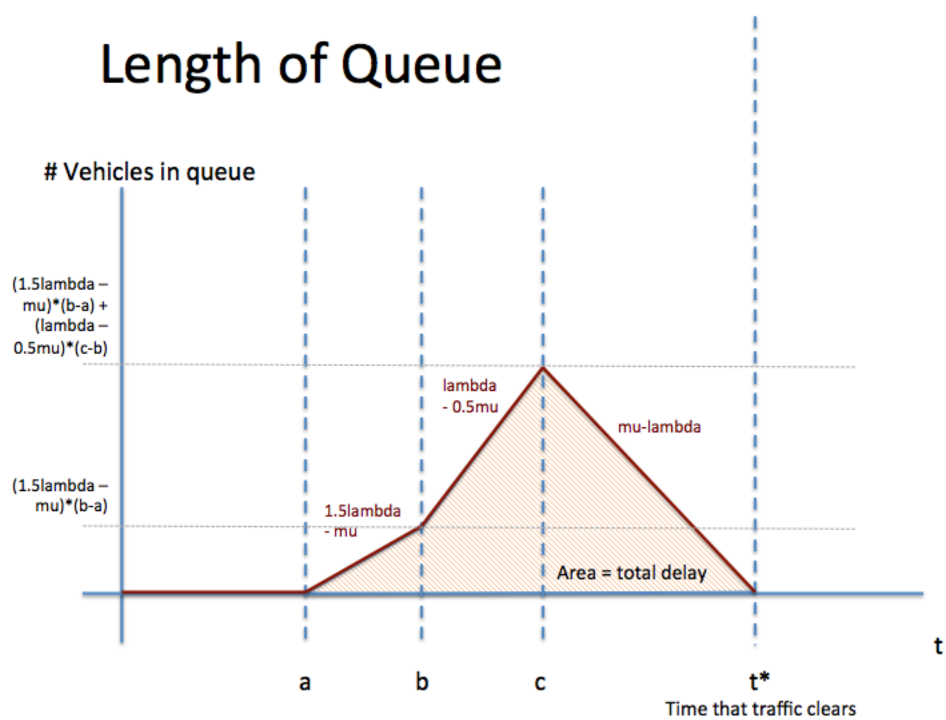
Consider the following figure. It shows the demand rate and the capacity of a roadway. Cars generally access the road at a rate λ. As is typical during rush hour, there is an increased rate of people using the roadway. In this example, this higher rate is 1.5λ and occurs from time $t = a$ to $t = b$. From time $t = b$ to $t = c$, an accident reduces the capacity to 0.5μ.



(a) Draw the Cumulative Diagram.



The other diagram that can be useful for calculations is the length of queue versus time.



(b) At what time does the traffic clear?

Solution Looking both diagrams, we can find the time t^* that the arrivals ($A(t)$) and departures ($D(t)$) meet, or when the length of queue hits zero. From the length of queue diagram:

$$(t^* - c)(\mu - \lambda) = (1.5\lambda - \mu)(b - a) + (\lambda - 0.5\mu)(c - b)$$
$$t^* = \frac{(1.5\lambda - \mu)(b - a) + (\lambda - 0.5\mu)(c - b)}{\mu - \lambda} + c$$

(c) What is the total delay experienced by all vehicles that experience delay?

Solution The total delay is the area between $A(t)$ and $D(t)$ in the cumulative diagram or the area under the length of queue diagram, which is

$$\frac{1}{2}(b-a)^2(1.5\lambda-\mu)+\frac{1}{2}(c-b)(2(1.5\lambda-\mu)(b-a)+(\lambda-0.5\mu)(c-b))+\frac{1}{2}(t^*-c)((1.5\lambda-\mu)(b-a)+(\lambda-0.5\mu)(c-b)).$$

(d) How many vehicles experience delay?

Solution We can find this by computing how many vehicles arrive during the period that there is a queue. That is, adding up the rate of arrival multiplied by the length of interval for each time interval:

$$(b - a)(1.5\lambda) + (t^* - b)\lambda$$

(e) Which vehicle has the longest delay? How long is the delay?

Solution The two candidates we should compare are 1) the one that arrives at time b and 2) the one that departs the queue at time c. The first candidate is the $a\lambda + 1.5(b - a)\lambda$-th vehicle who waits $\frac{2(1.5\lambda - \mu)(b - a)}{\mu}$. (We assume that $2(1.5\lambda - \mu)/\mu < (c - b)/(b - a)$.) The second candidate departing at time c is the $(a\lambda + \mu(b - a) + (c - b)(\mu/2))$-th vehicle. We can find his arrival time by looking at $A(t)$ and find out the x-axis (t) value when $A(t) = a\lambda + \mu(b - a) + (c - b)(\mu/2)$.

2 Runway Capacity at King's Landing

King's Landing has one small operating airport which could only serve medium (M) and light (L) aircraft until this Fall where heavy (H) aircraft like Boeing 787 can be routed here. Under ICAO's rules, the set of separations for arrivals on final approach are as given below:

Separation (nmiles)		Trailing Aircraft		
		H	M	L
Leading Aircraft	H	6	7	8
	M	4	5	6
	L	3	3	3

Table 1: Minimum separations for arrivals on final approach

Consider now a runway used only for arrivals at a major airport where ICAO separation standards are in use. Assume that the traffic mix at that airport and the aircraft characteristics are as follows:

The length of the final approach path, r, is 7 nautical miles.

Compute the arrivals capacity of this runway when $x = 10$, and compare with the capacity prior to the introduction of the heavy aircraft.

Aircraft Class	Approach Speed (knots)	Mix (%)	Runway Occupancy Time (ROT) on Landing (seconds)	Seats
H	150	x	70	320
M	135	$70 - x$	60	140
L	105	30	50	15

Table 2: Aircraft characteristics

Solution Two main components: the joint probability of two arrivals and the separation time between successive aircraft. The following table shows the minimum time separation between aircraft, which will be constant across the three scenarios.

$$T_{ij} = \max\left[\frac{r + s_{ij}}{v_j} - \frac{r}{v_i}, o_i\right], v_i > v_j$$

$$T_{ij} = \max\left[\frac{s_{ij}}{v_j}, o_i\right], v_i \leq v_j$$

Don't forget to convert between seconds and hours!

	H	M	L
H	144	205.33	346.29
M	96	133.33	259.05
L	72	80	102.86

Table 3: Minimum separation times between successive aircraft

Next, we calculate the probability of type i aircraft followed by type j aircraft for all i and j and each mix of H's.

	H	M	L
H	0	0	0
M	0	0.49	0.21
L	0	0.21	0.49

Table 4: Joint probability, $x = 0$

	H	M	L
H	0.01	0.06	0.03
M	0.06	0.36	0.18
L	0.03	0.18	0.09

Table 5: Joint probability, $x = 10$

From here, we can find the expected value of minimum inter-arrival time, which is equal to

$$E[T] = \sum_{i}^{K} \sum_{j}^{K} p_{ij} T_{ij}$$

x	$E[T]$	Arrivals/hr
0	150.3543	$23.94 \rightarrow 24$
10	145.7905	$24.69 \rightarrow 25$

Table 6: Expected value of inter-arrival times and arrival capacities

3 Traffic Flow Models

Greenshield The basic relationship for the Greenshield model: $v = v_{max}\left(1 - \dfrac{k}{k_{jam}}\right)$.
Suppose $v_{max} = 48$ km/h and $k_{jam} = 125$ veh/km. Plot the amount of time t it will take to travel these 3 km as a function of the flow q on the road segment.

Solution Because $q = vk$, we get

$$q = v_{max}k(1 - \frac{k}{k_{jam}})$$

Taking the derivative $dq/dk = 0$, we can find the critical density that gives us q_{max}. The critical density $k_c = k_{jam}/2$. Substituting this into the fundamental relationship, we can find the value of v associated with k_c. We then get

$$q_{max} = \frac{v_{max}k_{jam}}{4}.$$

For $v_{max} = 48$ km/h and $k_{jam} = 125$ veh/km, we have the following formula:

$$q = 48k - 0.384k^2, \qquad \text{for } 0 \leq k \leq 125.$$

For the road segment of length 3km, we start with the relationship $t = 3/v$. Since $k = q/v$, we also have $k = qt/3$ and we have:

$$q = 48\frac{qt}{3} - 0.384\left(\frac{qt}{3}\right)^2 16qt - 0.0427q^2t^2$$

Solving for q, we get

$$q = \frac{16t - 1}{0.0427t^2}.$$

Extended Free Flow Let k_j be the jam density of a single lane of a highway. The v vs. k relation is given as follows:

$$v = \begin{cases} v_f, & \text{if } k \leq k^* \qquad (1a) \\ c\left(\frac{k_j}{k} - 1\right), & \text{if } k \geq k^* \qquad (1b) \end{cases}$$

where v is in units of km/hour and k^* (the critical density), v_f (the "free flow speed"), c and k_j are all constants. Note that v has a single value at $k = k^*$, i.e., (1a) and (1b) give the same value of v when $k = k^*$.

We shall now consider the East-to-West (EW) lane of a two-lane rural highway. It has been observed that the EFF model provides a good approximation to the traffic flow characteristics of the EW lane. It has also been determined that, in this case, $q_{max} = 1600$ vehicles per hour (this is the maximum flow that the lane can support), $v_f = 80$ km/hour and $c = 20$.

(a) Please draw the q vs. k diagram for this lane of the rural highway.

Solution The fundamental relationship $(k - q)$ for the lane of the rural highway:

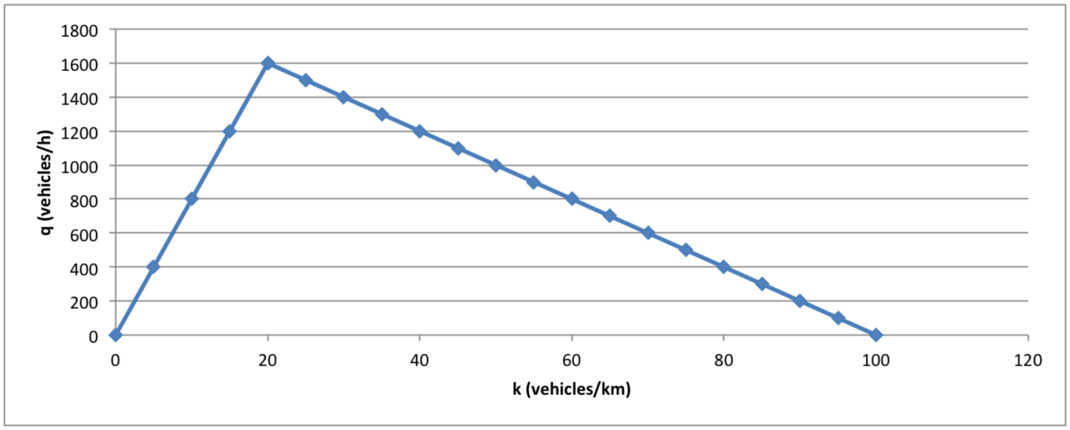
For $0 \leq k \leq k^*$,

$$q = ku = ku_f = 80k$$

For $k \geq k^*$,

$$q = ku = kc\left(\frac{k_j}{k} - 1\right) = 20(k_j - k)$$

The relationship is continuous, then we can find $q_{max} = 80k^*$. Given the value of q_{max}, we obtain $k^* = 1600/80 = 20$. We are certain that at $k = k^*$ gives the maximum q because u is a decreasing function in k after $k = k^*$. We can then find that $1600 = 20(k_j - 20)$. Therefore, $k_j = 100$ vehicles/km.



(b) Consider a 2-kilometer stretch of this highway. Plot the t vs. q relationship for a car traveling on one of the lanes of this highway for all the possible values of q. In other words, show on the vertical axis the amount of time it will take to travel this 2-kilometer segment for all possible values of q. Is t a single-valued function of q?

Solution To determine the plot of time compared to the flow, we first derive the density as a function of speed. We also plug in the formula $u = \frac{d}{t}$ to get density as a function of time. We then plug this into $q = uk$ to get flow as a function of time.

The relationship between t and q can be found from:

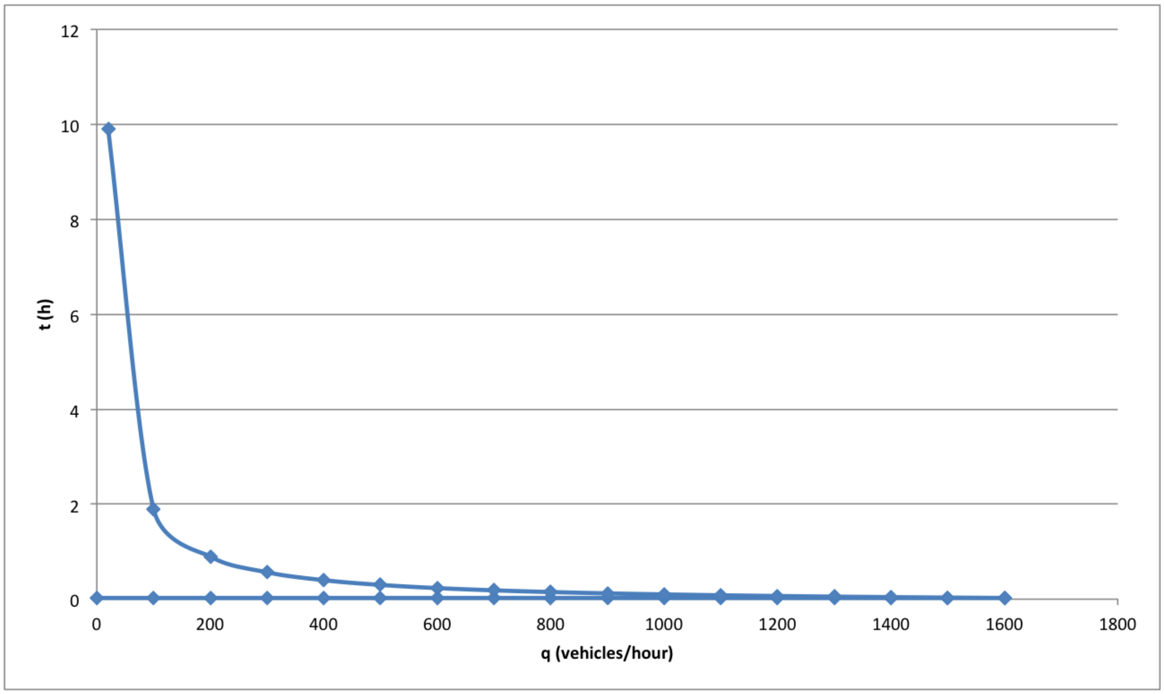
$$t = \frac{\text{distance}}{u} = \frac{2}{q/k} = \frac{2k}{q}$$

For $0 \leq k \leq k^*$:

$$t = \frac{2k}{u_f k} = \frac{2}{u_f} = 0.025.$$

For $k^* \leq k \leq k_j$:

$$\begin{aligned}
u = \frac{\text{distance}}{time} &= 20\left(\frac{100}{k} - 1\right) \\
\frac{2}{t} &= 20\left(\frac{100-k}{k}\right) \\
k &= 10t(100-k) \\
k(1+10t) &= 1000t \\
k &= \frac{1000t}{10t+1} \\
q = uk &= 2000 - 20k \\
q &= 2000 - \frac{20000t}{10t+1} \\
t &= \frac{200}{q} - \frac{1}{10}
\end{aligned}$$



t is not a single-valued function of q (as we have seen from the relationship between them and the figures above) because the same flow rate q can be achieved at two different times t and densities k.